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Cone Conditions and Properties of Sobolev Spaces

R. A. ADAMS AND JOHN FOURNIER*

*Department of Mathematics, The University of British Columbia,
Vancouver, B. C., Canada V6T 1W5*

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The Sobolev imbedding theorem and certain interpolation inequalities for Sobolev spaces are established for a wider class of domains than has been covered by earlier proofs. This class is defined by a weakened, measure theoretic version of the cone condition. The proofs are elementary.

1. INTRODUCTION

Many of the most common and useful properties of Sobolev spaces defined over a domain (open set) in Euclidean space require that the domain has a minimal degree of regularity. To this end, the domain is often assumed to satisfy a “cone condition.” For example, various imbeddings of Sobolev spaces into Lebesgue spaces or spaces of bounded continuous functions (the Sobolev imbedding theorem), and various interpolation inequalities such as those estimating L^p -norms of intermediate-order derivatives of functions in terms of such norms of higher- and lower-order derivatives (the Ehrling–Nirenberg–Gagliardo theorem), are commonly proved under the assumption that the domain satisfies a cone condition.

Several versions of the cone condition have been used, but the most common (and weakest) is as follows: The domain $\Omega \subset \mathbb{R}^n$ is said to satisfy *the cone condition* if each point $x \in \Omega$ is the vertex of a finite, right-spherical cone C_x contained in Ω and congruent to a fixed such cone C . (C_x is the union of all points on line segments from x to points of a ball not containing x .)

Many proofs based on the cone condition depend heavily on geometric consequences of the condition—for example the consequence that Ω is a finite union of subdomains each of which is a union of parallel translates of a parallelepiped. It has been noticed, however, (see Edmunds and Evans [4]), that in certain arguments of a potential theoretic nature an obviously weaker measure theoretic version of the cone condition suffices.

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Given x in Ω , let $R(x)$ consist of all points y in Ω such that the line segment joining x to y lies entirely in Ω ; thus $R(x)$ is a union of rays and line segments emanating from x . Let

$$\Gamma(x) = \{y \in R(x) : |y - x| < 1\},$$

and let $\mu_n(\Gamma(x))$ denote the Lebesgue measure of $\Gamma(x)$. We say that Ω satisfies the *weak cone condition* if there exists a number $\delta > 0$ such that

$$\mu_n(\Gamma(x)) \geq \delta \quad \text{for all } x \in \Omega.$$

Clearly, the cone condition implies the weak cone condition, and there are many domains satisfying the latter but not the former.

It is our purpose in this somewhat expository paper to show that the weak cone condition implies most of the standard imbedding and interpolation properties of ordinary Sobolev spaces. Furthermore, the proofs are easy. Previous proofs of these results have been based on the cone condition or on other even stronger regularity assumptions. Having in mind applications to nonlinear differential equations, some authors have used the theory of fractional-order Sobolev spaces (which requires quite regular domains) to justify certain interpolation inequalities for integral-order spaces. We obtain these inequalities by direct, elementary means for domains satisfying the weak cone condition.

We base some, but not all, of our results on potential theoretic arguments. It is seen that other methods can also be used effectively with the weak cone condition. All of our results are stated in Section 2 and the proofs given in subsequent sections.

2. SOBOLEV SPACES, IMBEDDINGS, AND INTERPOLATION INEQUALITIES

Let Ω be a domain in \mathbb{R}^n . We denote the norm in $L^p(\Omega)$ by $\|\cdot\|_{p,\Omega}$, omitting the domain from the symbol whenever confusion is unlikely to occur:

$$\begin{aligned} \|u\|_p &= \|u\|_{p,\Omega} = \left\{ \int_{\Omega} |u(x)|^p dx \right\}^{1/p}, \quad 1 \leq p < \infty, \\ \|u\|_{\infty} &= \|u\|_{\infty,\Omega} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|. \end{aligned}$$

For integral $m \geq 1$ and real $p \geq 1$ the Sobolev space $W^{m,p}(\Omega)$ consists of (equivalence classes of) functions $u \in L^p(\Omega)$ whose distributional derivatives $D^{\alpha}u$ of orders $|\alpha| \leq m$ also belong to $L^p(\Omega)$. (Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of nonnegative integers: $|\alpha| = \alpha_1 + \dots + \alpha_n$; $D^{\alpha} = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.) $W^{m,p}(\Omega)$ is a Banach space with norm

$$\|u\|_{m,p} = \|u\|_{m,p,\Omega} = \left\{ \sum_{|\alpha| \leq m} \|D^{\alpha}u\|_p^p \right\}^{1/p}.$$

The intersection with $W^{m,p}(\Omega)$ of the space $C^\infty(\Omega)$ of functions infinitely differentiable on Ω is dense in $W^{m,p}(\Omega)$. (See Meyers and Serrin [7] or Adams [1, p. 52].)

For integral $j \geq 0$ we denote by $C^j B(\Omega)$ the Banach space of functions u which possess on Ω bounded, continuous partial derivatives $D^\alpha u$ for $0 \leq |\alpha| \leq j$. The norm on $C^j B(\Omega)$ is

$$\|u\|_{C^j B(\Omega)} = \max_{|\alpha| \leq j} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

We are concerned with imbeddings (continuous injections) of $W^{m,p}(\Omega)$ into the spaces $C^j B(\Omega)$, $L^q(\Omega)$, and $L^q(\Omega \cap H)$ where H is a k -dimensional plane in \mathbb{R}^n . We write $W^{m,p}(\Omega) \rightarrow X$ to denote the imbedding of $W^{m,p}(\Omega)$ into the Banach space X and take this imbedding to be equivalent to the existence of a finite constant K such that for every $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$,

$$\|u\|_X \leq K \|u\|_{m,p}; \quad (1)$$

K is called the imbedding constant. This interpretation is justified since every element of $W^{m,p}(\Omega)$ is a norm limit in that space of a sequence of C^∞ functions which is, by virtue of (1), a Cauchy sequence in X and therefore convergent. The interpretation also obviates the difficulty which arises since elements of $W^{m,p}(\Omega)$ are really equivalence classes of functions equal a.e. on Ω , and cannot, strictly speaking, be said to belong to $C^j B(\Omega)$, or to $L^q(\Omega \cap H)$, if $\dim H = k < n$.

Our first goal is to establish the following version of the well-known Sobolev imbedding theorem.

THEOREM 1. *Let Ω be a domain in \mathbb{R}^n satisfying the weak cone condition. Let H be a k -dimensional plane in \mathbb{R}^n with $1 \leq k \leq n$. (If $k = n$ then $H = \mathbb{R}^n$.)*

Case I. If either $mp > n$ or $m = n$ and $p = 1$ then

$$W^{m+j,p}(\Omega) \rightarrow C^j B(\Omega) \quad \text{for } j \geq 0.$$

Moreover,

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega \cap H) \quad \text{for } p \leq q \leq \infty.$$

Case II. If $mp = n$ then

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega \cap H) \quad \text{for } p \leq q < \infty,$$

and in particular

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad \text{for } p \leq q < \infty.$$

Case III. If $mp < n$ and either $n - mp < k \leq n$, or $p = 1$ and $n - m \leq k \leq n$, then

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega \cap H) \quad \text{for } p \leq q \leq p^* = kp/(n - mp).$$

In particular,

$$W^{m,p}(\Omega) \rightarrow L^q(\Omega) \quad \text{for } p \leq q \leq p^* = np/(n - mp).$$

The imbedding constants for all of the above imbeddings depend only on m, n, p, q, j, k , and the constant δ of the weak cone condition.

Remarks 1. Many results of a related nature require the Sobolev imbedding theorem in their proofs, and rely on the cone condition only for this application. Therefore they remain valid under the weak cone condition. Examples (see Adams [1]) are

(a) the Rellich–Kondrachov theorem asserting the compactness of certain imbeddings of $W^{m,p}(\Omega)$ if Ω is bounded;

(b) the closure of $W^{m,p}(\Omega)$ under pointwise multiplication of its elements, provided $mp > n$;

(c) the analog of the Sobolev imbedding theorem for Orlicz–Sobolev spaces. (See Donaldson and Trudinger [3], or Adams [1, 2].)

Certain other results related to the imbedding theorem are proved by variations of the arguments used in the proof of Theorem 1, and the weak cone condition is sufficient for some of these. For example, if $mp = n$, there exists a sharp imbedding,

$$W^{m,p}(\Omega) \rightarrow L_A(\Omega),$$

where $L_A(\Omega)$ is an Orlicz space with defining N -function $A(t)$ equivalent near infinity to $e^{t^{n/(n-m)}} - 1$. (See [1, 12], or [4].)

2. It is clearly sufficient to prove Case I with $j = 0$. Case I is local in nature in that it supplies pointwise bounds for functions in terms of their $W^{m,p}$ -norms. It is proved directly for general m by potential arguments. In contrast we note that, if $k = n$, the proofs of Cases II and III can be reduced to consideration of the special case $m = p = 1$. (See Section 5.)

3. Certain refinements of Case I involving imbeddings of $W^{m,p}(\Omega)$ into spaces of uniformly continuous and Hölder continuous functions (see, for example, Adams [1]), require even more regularity of Ω than is afforded by the cone condition, and so cannot be proved using only the weak cone condition. However, some very useful refinements of the imbedding inequalities for all three cases can be derived using only the weak cone condition; these are given in Theorems 3 and 4 below.

4. A domain $\Omega \subset \mathbb{R}^n$ is said to have the *total extension* property if there exists a linear operator E mapping $C^\alpha(\Omega)$ into $C^\alpha(\mathbb{R}^n)$, and for each m and p a finite constant $K = K(m, p, \Omega)$, such that, for all $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$, we have

$$(i) \quad \|Eu\|_{m,p,\mathbb{R}^n} \leq K \|u\|_{m,p,\Omega},$$

and

$$(ii) \quad Eu(x) = u(x) \quad \text{if } x \in \Omega.$$

Many imbeddings of $W^{m,p}(\Omega)$ may be obtained relatively easily if $\Omega = \mathbb{R}^n$; these must then also hold for any domain Ω having the total extension property. This condition is, however, more restrictive than the cone condition, because it requires the domain to lie on one side of its boundary. It is nevertheless interesting to note that the weakest condition on Ω known to imply that Ω has the total extension property (see Stein [11, p. 189]), is also the weakest condition known to yield Hölder-continuity estimates for functions in $W^{m,p}(\Omega)$ with $mp > n$. (The condition is sometimes called the strong local Lipschitz condition.)

Our second goal in this paper is to prove three interpolation theorems under the weak cone condition. The first, Theorem 2, is a well-known result often associated with the names of Ehrling [5] and Nirenberg [8], and proved for domains satisfying the cone condition by Gagliardo [6]. The remaining two, Theorems 3 and 4, provide very sharp L^q estimates for functions in $W^{m,p}(\Omega)$. Some of these estimates can be obtained for regular domains via generalizations of the Sobolev imbedding theorem to Sobolev spaces of fractional order. We, however, obtain these estimates by elementary means without any reference to fractional-order spaces.

For integral $j \geq 0$ we define the seminorm $|u|_{j,p}$ by

$$|u|_{j,p} = \left\{ \sum_{|\alpha|=j} \|D^\alpha u\|_p^p \right\}^{1/p}.$$

THEOREM 2. *Let Ω be a domain in \mathbb{R}^n satisfying the weak cone condition. For each $\epsilon_0 > 0$, there exist constants $K = K(m, p, n, \epsilon_0, \delta)$ and $K' = K'(m, p, n, \epsilon_0, \delta)$ such that if $0 < \epsilon \leq \epsilon_0$ and $0 \leq j \leq m-1$ and $u \in W^{m,p}(\Omega)$, then*

$$\begin{aligned} |u|_{j,p} &\leq K\{\epsilon |u|_{m,p} + \epsilon^{-j/(m-j)} \|u\|_p\}, \\ \|u\|_{j,p} &\leq K'\{\epsilon \|u\|_{m,p} + \epsilon^{-j/(m-j)} \|u\|_p\}, \\ \|u\|_{j,p} &\leq 2K'(m, p, n, 1, \delta) \|u\|_{m,p}^{j/m} \|u\|_p^{(m-j)/m}. \end{aligned}$$

Remark. The second inequality follows from repeated applications of the first, and the third by setting $\epsilon_0 = 1$ and choosing ϵ in the second so that the two terms on the right side are equal. Therefore, only the first inequality requires proof.

THEOREM 3. Let Ω be a domain in \mathbb{R}^n satisfying the weak cone condition. If $mp > n$, let $p \leq q \leq \infty$; if $mp = n$, let $p \leq q < \infty$; if $mp < n$, let $p \leq q \leq p^* = np/(n - mp)$. Then there exists a constant $K = K(m, n, p, q, \delta)$ such that, for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_q \leq K \|u\|_{m,p}^\theta \|u\|_p^{1-\theta},$$

where $\theta = (n/mp) - (n/mq)$.

A special case of Theorem 3 asserts that, if $mp > n$, then

$$\|u\|_\infty \leq K \|u\|_{m,p}^{n/mp} \|u\|_p^{1-(n/mp)}.$$

A similar inequality with $\|u\|_p$ replaced by a more general $\|u\|_q$ is sometimes useful, and is given in Theorem 4.

THEOREM 4. Let Ω be a domain in \mathbb{R}^n satisfying the weak cone condition. Let $q \geq 1$, and $p > 1$. Suppose that $mp - p < n < mp$. Then there exists a constant $K = K(m, n, p, q, \delta)$ such that, for every $u \in W^{m,p}(\Omega)$,

$$\|u\|_\infty \leq K \|u\|_{m,p}^\theta \|u\|_q^{1-\theta},$$

where $\theta = np/[np + (mp - n)q]$.

Case I of Theorem 1 is proved in Section 3 below. Cases II and III, for $p > 1$, are treated in Section 4 by potential theoretic arguments similar to those in Stein's book [11]. The potential method is very simple, and, modulo required applications of the Marcinkiewicz interpolation theorem, elementary; but it gives incomplete results if $p = 1$. In Section 5 Cases II and III are established for $p = 1$ by a combinatorial-averaging argument having its roots in the combinatorial method used by Gagliardo in [6]. (Certain cases with $k < n$, $p = 1$ are not dealt with directly by the alternative method, but converted to situations where $p > 1$, and the results of Section 4 applied.) It is also shown that the cases when $k = n$ and $p > 1$ can always be reduced to the case $p = 1$.

The interpolation Theorems 2, 3, and 4 are proved in Section 6.

3. LOCAL ESTIMATES

We begin by preparing two lemmas for immediate and future use. Let γ denote Lebesgue surface measure on the unit sphere $\Sigma = \{\sigma \in \mathbb{R}^n: |\sigma| = 1\}$

LEMMA 1. Let the domain $\Omega \subset \mathbb{R}^n$ satisfy the weak cone condition. Then there exist positive constants $\eta \leq 1$, A , and B depending on n and δ , and for each $x \in \Omega$ a subset $P_{x,\eta} \subset \Sigma$, such that $\lambda(P_{x,\eta}) = A$ and $x + t\sigma \in \Omega$ if $\sigma \in P_{x,\eta}$ and

$0 < t < \eta$. In particular, for each $x \in \Omega$ and each ξ satisfying $0 < \xi \leq \eta$, the "generalized cone" $C_{x,\xi} = \{y = x + t\sigma \in \mathbb{R}^n; \sigma \in P_{x,\eta}, 0 < t < \xi\}$ satisfies $C_{x,\xi} \subset \Omega$ and $\mu_n(C_{x,\xi}) = B_\xi^n$.

Proof. Let η be the radius of the ball of volume $\delta/2$ in \mathbb{R}^n , and let $B_\eta(x)$ denote the ball of radius η centered at x . Then for each $x \in \Omega$ we have $\mu_n(\Gamma(x) \sim B_\eta(x)) \geq \delta/2$. The radial projection of $\Gamma(x) \sim B_\eta(x)$ onto the sphere $x + \Sigma$ has surface measure not less than $n\delta/2 = A$, and the result follows at once. ■

LEMMA 2. Let the domain $\Omega \subset \mathbb{R}^n$ satisfy the weak cone condition. There exists a constant $K_1 = K_1(n, m, \delta)$ such that for every $u \in C^\infty(\Omega)$, every $x \in \Omega$, and every ξ satisfying $0 < \xi \leq \eta$,

$$|u(x)| \leq K_1 \left\{ \sum_{|\alpha| \leq m-1} \xi^{|\alpha|-n} \int_{C_{x,\xi}} |D^\alpha u(y)| dy + \sum_{|\alpha|=m} \int_{C_{x,\xi}} |D^\alpha u(y)| |x-y|^{m-n} dy \right\}, \quad (2)$$

where $C_{x,\xi}$ is the generalized cone of Lemma 1.

Proof. For $y \in C_{x,\xi}$ we apply Taylor's formula

$$f(1) = \sum_{j=0}^{m-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(t) dt$$

to the function $f(t) = u(tx + (1-t)y)$ and, noting that

$$f^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} D^\alpha u(tx + (1-t)y)(x-y)^\alpha,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $(x-y)^\alpha = (x_1-y_1)^{\alpha_1} \cdots (x_n-y_n)^{\alpha_n}$, we obtain

$$|u(x)| \leq \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} |D^\alpha u(y)| |x-y|^{|\alpha|} + \sum_{|\alpha|=m} \frac{m}{\alpha!} |x-y|^m \int_0^1 (1-t)^{m-1} |D^\alpha u(tx + (1-t)y)| dt.$$

[Integration of y over $C_{x,\xi}$ leads to

$$B_\xi^n |u(x)| \leq \sum_{|\alpha| \leq m-1} \frac{\xi^{|\alpha|}}{\alpha!} \int_{C_{x,\xi}} |D^\alpha u(y)| dy + \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_{C_{x,\xi}} |x-y|^m dy \int_0^1 (1-t)^{m-1} |D^\alpha u(tx + (1-t)y)| dt.$$

We interchange the order of the double integral and substitute $z = tx + (1-t)y$ to obtain, for that integral,

$$\begin{aligned} & \int_0^1 (1-t)^{-n-1} dt \int_{C_{x,(1-t)\xi}} |x-z|^m |D^\alpha u(z)| dz \\ &= \int_{C_{x,\xi}} |x-z|^m |D^\alpha u(z)| dz \int_0^{1-|x-z|/\xi} (1-t)^{-n-1} dt \\ &\leq \frac{\xi^n}{n} \int_{C_{x,\xi}} |D^\alpha u(z)| |x-z|^{m-n} dz. \end{aligned}$$

Inequality (2) is now immediate. ■

Proof of Theorem 1, Case I. We must show that (assuming $j = 0$)

$$|u(x)| \leq K \|u\|_{m,p} \quad (3)$$

holds for all $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$ and all $x \in \Omega$. If $p = 1$ and $m = n$, (3) follows immediately from (2) with $\xi = \eta$. If $p > 1$ and $mp > n$ we apply Hölder's inequality to (2) (with $\xi = \eta$) to obtain

$$\begin{aligned} |u(x)| &\leq \left\{ \sum_{|\alpha| \leq m-1} B^{(p-1)/p} \eta^{|\alpha|-(n/p)} \|D^\alpha u\|_{p,C_{x,\eta}} \right. \\ &\quad \left. + \sum_{|\alpha|=m} \|D^\alpha u\|_{p,C_{x,\eta}} \left(\int_{C_{x,\eta}} |x-y|^{(m-n)p/(p-1)} dy \right)^{(p-1)/p} \right\} \end{aligned}$$

which gives (3) since $C_{x,\eta} \subset \Omega$ and the last integral is finite. ■

4. L^q IMBEDDINGS BY POTENTIAL ARGUMENTS

Except in Lemma 3 we assume throughout this section that $p > 1$, and we denote $p' = p/(p-1)$. Let 1_{B_r} denote the characteristic function of the ball $B_r = B_r(0)$ having center 0 and radius r in \mathbb{R}^n . Let $B_1 = B$, $1_{B_1} = 1_B$, $\omega_m(x) = |x|^{m-n}$, and $1_{B_r} \omega_m(x) = 1_{B_r}(x) \omega_m(x)$. Clearly $1_B(x) \leq 1_{B\omega_m}(x) \leq \omega_m(x)$ for all $x \in \mathbb{R}^n$ if $m \leq n$.

The operator of convolution with ω_m is called a "fractional" integral of order m . Such operators have been studied by many authors; indeed, Sobolev [9] based his proof of the imbedding theorem for $\Omega = \mathbb{R}^n$ and $k = n$ on properties of these operators. The following two lemmas are known; we give proofs here because, when $k < n$, our proofs are simpler than those given elsewhere. The arguments are adaptations of those given in the books by Sobolev [10, p. 43], and Stein [11, p. 119].

LEMMA 3. Let $p \geq 1$, $mp \leq n$, and $n - mp < k \leq n$. Then there exists a constant $K_2 = K_2(m, p, n, k)$ such that for every $r > 0$, for every k -plane H in \mathbb{R}^n , and for every $v \in L^p(\mathbb{R}^n)$, the convolution $1_{B_r} \omega_m * |v|$ has a trace on H belonging to $L^p(H)$, and

$$\|1_{B_r} \omega_m * |v|\|_{p, H} \leq K_2 r^{m-(n-k)/p} \|v\|_{p, \mathbb{R}^n}. \quad (4)$$

In particular

$$\|1_B * |v|\|_{p, H} \leq \|1_B \omega_m * |v|\|_{p, H} \leq K_2 \|v\|_{p, \mathbb{R}^n}.$$

Proof. By Hölder's inequality (if $p > 1$)

$$\begin{aligned} & 1_{B_r} \omega_m * |v|(x) \\ &= \int_{B_r(x)} |v(y)| |x - y|^{-\beta} |x - y|^{\beta+m-n} dy \\ &\leq \left\{ \int_{B_r(x)} |v(y)|^p |x - y|^{-\beta p} dy \right\}^{1/p} \left\{ \int_{B_r(x)} |x - y|^{(\beta+m-n)p'} dy \right\}^{1/p'} \\ &= K_3(m, p, n, \beta) r^{\beta+m-(n/p)} \left\{ \int_{B_r(x)} |v(y)|^p |x - y|^{-\beta p} dy \right\}^{1/p} \end{aligned}$$

provided $\beta + m - (n/p) > 0$. If $p = 1$ the same estimate holds provided $\beta + m - n \geq 0$ without the use of Hölder's inequality. If dx' denotes the Lebesgue volume element in H , then

$$\begin{aligned} \int_H dx' \int_{B_r(x)} |v(y)|^p |x - y|^{-\beta p} &\leq \int_{\mathbb{R}^n} |v(y)|^p dy \int_{H \cap B_r(y)} |x - y|^{-\beta p} dx' \\ &\leq K_4(p, k, \beta) r^{k-\beta p} \|v\|_{p, \mathbb{R}^n}^p, \end{aligned}$$

provided $k > \beta p$. Since $n - mp < k$ we can choose $\beta = \beta(m, p, n, k)$ so that $k > \beta p$ and $\beta + m - (n/p) > 0$. Hence

$$\int_H |1_{B_r} \omega_m * |v||^p(x) dx' \leq K_3^p K_4 r^{mp-n+k} \|v\|_{p, \mathbb{R}^n}^p$$

as required. ■

In the following lemma the notion of weak type operator and the Marcinkiewicz interpolation theorem are required. (See, for example, Stein [11, Appendix B].)

A linear operator I taking functions defined (a.e.) on \mathbb{R}^n into functions defined (a.e.) on \mathbb{R}^k is said to be of weak type (p, q) , ($1 \leq p, q \leq \infty$) if there exists a constant K such that for each $\lambda > 0$ and each $v \in L^p(\mathbb{R}^n)$

$$\mu_k(\{x \in \mathbb{R}^k : |Iv(x)| > \lambda\}) \leq [(K/\lambda) \|v\|_{p, \mathbb{R}^n}]^q \quad (\text{if } q < \infty)$$

or

$$\|Iv\|_{\infty, \mathbb{R}^k} \leq K \|v\|_{p, \mathbb{R}^n} \quad (\text{if } q = \infty).$$

Evidently any bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^k)$ is of weak type (p, q) . The Marcinkiewicz interpolation theorem asserts that, if I is simultaneously of weak types (p_1, q_1) with constant K_1 and (p_2, q_2) with constant K_2 , if $p_1 \leq q_1$ and $p_2 \leq q_2$, and if

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

for some θ , $0 < \theta < 1$, then there exists a constant $K = K(p_1, p_2, q_1, q_2, \theta, K_1, K_2)$ such that for all $v \in L^p(\mathbb{R}^n)$ we have $Iv \in L^q(\mathbb{R}^k)$ and

$$\|Iv\|_{q, \mathbb{R}^k} \leq K \|v\|_{p, \mathbb{R}^n}.$$

LEMMA 4. Let $1 < p$, $mp < n$, $n - mp < k \leq n$ and $p^* = kp/(n - mp)$. Then there exists a constant $K_5 = K_5(m, p, n, k)$ such that for every k -plane H in \mathbb{R}^n , and every $v \in L^p(\mathbb{R}^n)$, the convolution $\omega_m * |v|$ has a trace on H belonging to $L^{p^*}(H)$ and

$$\|1_H * |v|\|_{p^*, H} \leq \|1_B \omega_m * |v|\|_{p^*, H} \leq \|\omega_m * |v|\|_{p^*, H} \leq K_5 \|v\|_{p, \mathbb{R}^n}. \quad (5)$$

Proof. Since $mp < n$ we have for each $x \in \mathbb{R}^n$, by Hölder's inequality, that

$$\int_{\mathbb{R}^n \sim B_r(x)} |v(y)| |x - y|^{m-n} dy \leq K_6(m, p, n) r^{m-(n/p)} \|v\|_{p, \mathbb{R}^n}.$$

Let $\lambda > 0$ and choose r so that $K_6 r^{m-(n/p)} \|v\|_{p, \mathbb{R}^n} = \lambda/2$. Then

$$\begin{aligned} \mu_k(\{x \in H : \omega_m * |v|(x) > \lambda\}) &\leq \mu_k(\{x \in H : 1_{B_r} \omega_m * |v|(x) > \lambda/2\}) \\ &\leq (\lambda/2)^{-p} \|1_{B_r} \omega_m * |v|\|_{p, H}^p \\ &\leq (K_2/K_6)^p r^k = (K_2/K_6)^p ((2K_6/\lambda) \|v\|_{p, \mathbb{R}^n})^{p^*}, \end{aligned}$$

where we have used (4) to obtain the last inequality. Hence the mapping $v \rightarrow (\omega_m * |v|)|_H$ is of weak type (p, p^*) . For fixed n , m , and k , the values of p satisfying the conditions of the lemma constitute an open interval. Since $p^* > p$ and since $1/p^*$ is an affine function of $1/p$ the Marcinkiewicz theorem yields $K_5 = K_5(m, p, n, k)$ such that the final inequality of (5) holds. The other inequalities are trivial. ■

Proof of Theorem 1, Case III, for $p > 1$. We have $mp < n$, $n - mp < k \leq n$ and $p^* = kp/(n - mp)$. Let $u \in C^\infty(\Omega)$ and extend u and all derivatives D^α

to all of \mathbb{R}^n to vanish identically outside of Ω . (Thus L^p -norms of these functions taken over \mathbb{R}^n are identical to the same norms taken over Ω .) Taking $\xi = \eta$ in Lemma 2 and replacing $C_{x,\eta}$ on the right side of (2) with the larger set B , and K_1 with $K_1\eta^{-n}$ we obtain

$$|u(x)| \leq K_1 \left\{ \sum_{|\alpha| \leq m-1} 1_B * |D^\alpha u|(x) + \sum_{|\alpha|=m} 1_B \omega_m * |D^\alpha u|(x) \right\}. \quad (6)$$

Let $K_7 = K_7(m, n)$ be the number of multi-indices α of order $|\alpha| \leq m$. If $p \leq q \leq p^*$ we set $1/q = (\theta/p) + (1-\theta)/p^*$ and obtain by Hölder's inequality and Lemmas 3 and 4

$$\begin{aligned} \|u\|_{q, \Omega \cap H} &\leq \|u\|_{p, H}^\theta \|u\|_{p^*, H}^{1-\theta} \\ &\leq (K_1 K_2 K_7 \|u\|_{m, p, \Omega})^\theta (K_1 K_5 K_7 \|u\|_{m, p, \Omega})^{1-\theta} \\ &= K \|u\|_{m, p, \Omega} \end{aligned}$$

as required. ■

Proof of Theorem 1, Case II, for $p > 1$. We have $p > 1$, $mp = n$, and $p \leq q < \infty$. We may select numbers p_1, p_2 , and θ so that $1 < p_1 < p < p_2$, $n - mp_1 < k$, $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{p_1}.$$

As in the proof of Case III the maps $v \rightarrow (1_B * |v|)|_H$ and $v \rightarrow (1_B \omega_m * |v|)|_H$ are bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_1}(\mathbb{R}^k)$ and so of weak type (p_1, p_1) with the constant depending only on n, m, p_1 , and k . As in the proof of Case I these same maps are bounded from $L^{p_2}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^k)$ and so are of weak type (p_2, ∞) , again with constant depending on n, m, p_2 , and k . By the Marcinkiewicz theorem there exists $K_8 = K_8(m, p, n, q, k)$ such that

$$\|1_B * |v|\|_{q, H} \leq \|1_B \omega_m * |v|\|_{q, H} \leq K_8 \|v\|_{p, \mathbb{R}^n}$$

and the desired result follows on application of these estimates to the various terms of (6). ■

Remark. For $k = n$ this case can also be proved by direct application of Young's inequality for convolutions to the terms of (6).

5. L^q IMBEDDINGS BY AVERAGING

The averaging method of this section is based on the following combinatorial lemma which generalizes a result of Gagliardo [6].

LEMMA 5. Let Ω be a domain in \mathbb{R}^n , where $n \geq 2$, and let $\sigma_1, \sigma_2, \dots, \sigma_n$ be linearly independent unit vectors in \mathbb{R}^n . For each j , $1 \leq j \leq n$, let Ω_j be the orthogonal projection of Ω onto an $(n-1)$ -dimensional plane H_j normal to σ_j , and let u_j be a function on \mathbb{R}^n with the properties

(i) u_j is invariant under translation along σ_j ; i.e.,

$$u_j(x + t\sigma_j) = u_j(x) \quad \text{for all } t \in \mathbb{R}.$$

(ii) $u_j|_{\Omega_j} \in L^{n-1}(\Omega_j)$.

Then $u = \prod_{j=1}^n u_j \in L^1(\Omega)$ and

$$\|u\|_{1,\Omega} \leq (\det \sigma)^{-1/(n-1)} \prod_{j=1}^n \|u_j\|_{n-1,\Omega_j} \quad (7)$$

where σ is the matrix of components of $\sigma_1, \dots, \sigma_n$.

Proof. Without loss of generality we may assume $\Omega = \mathbb{R}^n$. (Redefining each u_j to be identically zero outside the cylinder with cross section Ω_j parallel to σ reduces the general case to this one.) If $\sigma_j = e_j$, the unit vector along the j th coordinate axis ($1 \leq j \leq n$) then u_j is independent of x_j and (7) becomes in this case

$$\left\{ \int_{\mathbb{R}^n} |u(x)| dx \right\}^{n-1} \leq \prod_{j=1}^n \int_{\mathbb{R}^{n-1}} |u_j(x)|^{n-1} d\hat{x}_j \quad (8)$$

where $d\hat{x}_j = dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n$. This special case can be proved by induction on n from the trivial case $n = 2$ by using Hölder's inequality. (The details can be found in Adams [1, p. 101] and in Gagliardo [6].)

For arbitrary σ_j we note that $\sigma e_j = \sigma_j$ ($1 \leq j \leq n$) where σ is the matrix with columns $\sigma_1, \dots, \sigma_n$. Let $x = \sigma y$ and set $v_j(y) = u_j(\sigma y) = u_j(x)$. Then for each j we have $v_j(y + te_j) = u_j(x + t\sigma_j) = u_j(x) = v_j(y)$ so that v_j is independent of y_j . Hence by (8),

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)| dx &= |\det \sigma| \int_{\mathbb{R}^n} \prod_{j=1}^n |v_j(y)| dy \\ &\leq |\det \sigma| \prod_{j=1}^n \left\{ \int_{\mathbb{R}^{n-1}} |v_j(y)|^{n-1} d\hat{y}_j \right\}^{1/(n-1)}. \end{aligned} \quad (9)$$

If ds_j denotes the Lebesgue volume element in the $(n-1)$ -plane P_j spanned by $\sigma_1, \dots, \sigma_{j-1}, \sigma_{j+1}, \dots, \sigma_n$ and δ_j is the $(n-1)$ -volume of the parallelepiped Q_j spanned by those vectors then

$$\int_{\mathbb{R}^{n-1}} |v_j(y)|^{n-1} d\hat{y}_j = \frac{1}{\delta_j} \int_{P_j} |u_j(x)|^{n-1} ds_j. \quad (10)$$

Since translation parallel to σ_j does not affect the value of u_j we have, further, that

$$\int_{P_j} |u_j(x)|^{n-1} ds_j = \frac{1}{\epsilon_j} \|u_j\|_{n-1, H_j}^{n-1} \quad (11)$$

where ϵ_j is the $(n-1)$ -volume of the projection of the parallelepiped Q_j onto H_j . But $\delta_j \epsilon_j = |\det \sigma|$ (the volume of the parallelepiped spanned by $\sigma_1, \dots, \sigma_n$), and, by combining (9), (10), and (11) we obtain (7). ■

The following two lemmas show that the proofs of Cases II and III of Theorem 1 for $k = n$ can always be reduced to consideration of the special case $m = p = 1$.

LEMMA 6. *Let Ω be a domain in \mathbb{R}^n . If $W^{1,r}(\Omega) \rightarrow L^q(\Omega)$ whenever either $r < n$ and $r \leq q \leq nr/(n-r)$, or $r = n$ and $r \leq q < \infty$, then $W^{m,p}(\Omega) \rightarrow L^q(\Omega)$ whenever either $mp < n$ and $p \leq q \leq np/(n-mp)$ or $mp = n$ and $p \leq q < \infty$.*

Proof. We proceed by induction on m . Assume the assertion holds for $m-1$ and let $u \in W^{m,p}(\Omega)$ where $mp \leq n$. Then u and $\partial u / \partial x_j$ ($1 \leq j \leq n$), belong to $W^{m-1,p}(\Omega)$ and so, by assumption, $u \in W^{1,r}(\Omega)$ where $p \leq r \leq np/(n-mp+p)$. Since $mp \leq n$, we have that $r \leq n$, and $W^{1,r}(\Omega) \rightarrow L^q(\Omega)$, where $p \leq r \leq q \leq nr/(n-r) = np/(n-mp)$ if $r < n$ (i.e., if $mp < n$), or where $p \leq r \leq q < \infty$ if $r = n$ (i.e., if $mp = n$). Hence $W^{m,p}(\Omega) \rightarrow L^q(\Omega)$ and the induction is complete. ■

LEMMA 7. *Let Ω be a domain in \mathbb{R}^n . If $W^{1,1}(\Omega) \rightarrow L^r(\Omega)$ for $1 \leq r \leq n/(n-1)$ then $W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ whenever $p < n$ and $p \leq q \leq np/(n-p)$ or $p = n$ and $p \leq q < \infty$.*

Proof. Fix q in the specified range and let $\lambda = 1 + (p-1)q/p$; then $p \leq \lambda \leq p(n-1)/(n-p)$ if $p < n$, and $p \leq \lambda < \infty$ if $p = n$. Let $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ and suppose, for the moment, that $\|u\|_q < \infty$. Let $v(x) = |u(x)|^\lambda$. Since $\partial |u(x)|^\lambda / \partial x_j = \lambda |u(x)|^{\lambda-1} \operatorname{sgn} u(x) (\partial u / \partial x_j)$ we have, by Hölder's inequality, that

$$\begin{aligned} \|v\|_{1,1} &= \int_{\Omega} |u(x)|^{\lambda-1} \left(|u(x)| + \lambda \sum_{j=1}^n \left| \frac{\partial}{\partial x_j} u(x) \right| \right) dx \\ &\leq K_1(n, p, q) \|u\|_q^{(p-1)q/p} \|u\|_{1,p} < \infty. \end{aligned}$$

Now $1 \leq q/\lambda \leq n/(n-1)$; therefore $v \in L^{q/\lambda}(\Omega)$ and

$$\|u\|_q^q = \|v\|_{q/\lambda}^{q/\lambda} \leq K_2 \|v\|_{1,1}^{q/\lambda} \leq K_3 \|u\|_q^{(p-1)q^2/\lambda p} \|u\|_{1,p}^{q/\lambda}.$$

It follows by cancellation that

$$\|u\|_q \leq K \|u\|_{1,p} \quad (12)$$

For positive integral k let $f_k \in C^\infty(\mathbb{R})$ satisfy

- (i) $f_k(t) = t$ for $|t| \leq k$,
- (ii) $f_k'(t) \leq 1$ for all t ,
- (iii) $f_k'(t) = 0$ for $|t| \geq k+1$.

If $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ is real-valued then $f_k \circ u$ satisfies $\|f_k \circ u\|_{1,p} \leq \|u\|_{1,p}$ and also $f_k \circ u \in C^\infty(\Omega) \cap L^q(\Omega)$. Thus (12) holds for $f_k \circ u$ and hence also for u by monotone convergence. Extension to complex-valued u follows from separate applications to the real and imaginary parts, and the proof is complete. ■

Proof of Theorem 1, Cases II and III, for $k = n$. By virtue of Lemmas 6 and 7 we may assume $m = p = 1$, and in view of Case I, that $n > 1$. Accordingly, let $p^* = n/(n-1)$.

Let the sets Σ and $P_{x,\eta}$, the measure λ , and the constants A and η be as specified in Lemma 1. For $x \in \Omega$ and $\sigma \in \Sigma$ let $\rho_{x,\sigma} = \min(1, \inf\{t > 0: x + t\sigma \notin \Omega\})$. By Lemma 1, we have that $\rho_{x,\sigma} \geq \eta$ for $\sigma \in P_{x,\eta}$, and also that $\lambda(P_{x,\eta}) = A$.

Let $u \in C^\infty(\Omega)$ and fix $x \in \Omega$ and $\sigma \in \Sigma$. Observe that, if $f \in C^1([0, \rho])$, then

$$|f(0)| \leq \frac{1}{\rho} \int_0^\rho (|f(t)| + |f'(t)|) dt.$$

Applying this observation with $f(t) = u(x + t\sigma)$ we see that

$$|u(x)| \leq \frac{1}{\rho_{x,\sigma}} \int_{\ell_{x,\sigma}} (|u| + |\text{grad } u|) ds, \quad (13)$$

where $\ell_{x,\sigma}$ denotes the intersection of Ω with the line through x parallel to σ , and ds is the length element on that line.

Let λ^n denote the product measure on Σ^n . Since the determinant map $\det: \Sigma^n \rightarrow \mathbb{R}$ satisfies $\lambda^n(\det^{-1}\{0\}) = 0$, there exists an integer $N = N(n, A)$ such that

$$\lambda^n(\det^{-1}[-1/N, 1/N]) < A^n/2.$$

Let $T = \{\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma^n: |\det \sigma| > 1/N\}$, and for each $x \in \Omega$ let $S(x) = \{\sigma \in T: \rho_{x,\sigma_j} \geq \eta \text{ for } 1 \leq j \leq n\}$. Then

$$\lambda^n(S(x)) > \prod_{j=1}^n \lambda(P_{x,\eta}) - \frac{A^n}{2} \geq \frac{A^n}{2}.$$

For each $\sigma \in \Sigma$, let

$$u_\sigma(x) = \left\{ \int_{L_{x,\sigma}} (|u| + |\operatorname{grad} u|) ds \right\}^{1/(n-1)}.$$

Clearly $u_\sigma(x + t\sigma) = u_\sigma(x)$ for all $t \in \mathbb{R}$. Moreover, if H_σ is a hyperplane perpendicular to σ , if ds_σ denotes the $(n-1)$ -volume element in H_σ , and if Ω_σ is the projection of Ω onto H_σ , then we have

$$\int_{\Omega_\sigma} |u_\sigma(x)|^{n-1} ds_\sigma \leq \|u\|_{1,1,\Omega}.$$

For $x \in \Omega$ and $\sigma = (\sigma_1, \dots, \sigma_n) \in S(x)$ we have by (13) that

$$|u(x)|^{n/(n-1)} \leq \eta^{-n/(n-1)} \prod_{j=1}^n u_{\sigma_j}(x).$$

We integrate σ over $S(x) \subset T$ to obtain

$$\lambda^n(S(x)) |u(x)|^{n/(n-1)} \leq K_1 \int_{S(x)} \prod_{j=1}^n u_{\sigma_j}(x) d\sigma,$$

where $K_1 = K_1(n, \delta) = \eta^{-n/(n-1)}$. Hence

$$|u(x)|^{n/(n-1)} \leq \frac{2K_1}{A^n} \int_T \prod_{j=1}^n u_{\sigma_j}(x) d\sigma.$$

Setting $K_2 = 2K_1/A^n$ and applying Lemma 5, we have

$$\begin{aligned} \int_\Omega |u(x)|^{n/(n-1)} dx &\leq K_2 \int_T d\sigma \int_\Omega \prod_{j=1}^n u_{\sigma_j}(x) dx \\ &\leq K_2 \int_T |\det \sigma|^{-1/(n-1)} d\sigma \prod_{j=1}^n \left\{ \int_{\Omega_{\sigma_j}} |u_{\sigma_j}(x)|^{n-1} ds_{\sigma_j} \right\}^{1/(n-1)} \\ &\leq K_2 N^{1/(n-1)} \lambda^n(\Sigma^n) \|u\|_{1,1,\Omega}^{n/(n-1)}, \end{aligned}$$

whence $\|u\|_{n/(n-1)} \leq K \|u\|_{1,1}$.

If $1 \leq q \leq n/(n-1)$ then $1/q = \theta + (1-\theta)(n-1)/n$ and, by Hölder's inequality, $\|u\|_q \leq \|u\|_1^\theta \|u\|_{n/(n-1)}^{1-\theta} \leq K^{1-\theta} \|u\|_{1,1}$. This completes the proof. ■

Completion of the Proof of Theorem 1. There remain to be proved two special cases of Theorem 1, both falling under Case III with $p = 1$ and $k < n$. These are

- (A) $p = 1, n > m \geq 2, n - m < k < n, p^* = k/(n - m),$
- (B) $p = 1, n > m, n - m = k < n, p^* = 1.$

We deal with situation (A) first. We know that $W^{m,1}(\Omega) \rightarrow W^{m-1,r}(\Omega)$ where $r = n/(n-1) > 1$. Since $k \geq n - m + 1$, we have $n - (m-1)r < k$, and so, by the proof given in Section 4,

$$W^{m-1,r}(\Omega) \rightarrow L^{p^*}(\Omega \cap H), \quad \text{for } p^* = kr/(n - mr + r) = k/(n - m).$$

Thus $W^{m,1}(\Omega) \rightarrow L^{p^*}(\Omega \cap H)$. By Lemmas 2 and 3, $W^{m,1}(\Omega) \rightarrow L^1(\Omega \cap H)$ as well. (The imbedding constants for both of these imbeddings depend only on m, n, k , and δ .) Finally, an application of Hölder's inequality shows that

$$W^{m,1}(\Omega) \rightarrow L^q(\Omega \cap H) \quad \text{if } 1 \leq q \leq p^*.$$

To treat situation (B) we use an averaging argument similar to that used in the proof of the case $k = n$ above, but without the combinatorial complications present in that proof. Let $\mathcal{O}(n)$ denote the orthogonal group in \mathbb{R}^n (the $n \times n$ orthogonal matrices) and let μ denote the Haar measure on $\mathcal{O}(n)$, normalized so that $\mu(\mathcal{O}(n)) = 1$. For each $\sigma \in \mathcal{O}(n)$ let σ_j be the j th column of σ ($\sigma e_j = \sigma_j, 1 \leq j \leq n$).

We assume, with no loss of generality, that the k -plane H is the plane $x_1 = x_2 = \dots = x_m = 0$ and write, whenever convenient, $x = (x', x'')$ where $x' = (x_1, \dots, x_m)$, $x'' = (x_{m+1}, \dots, x_n)$. For each $x = (0, x'') \in H$ and each $\sigma \in \mathcal{O}(n)$ let $E(\sigma, x)$ be the m -plane through x spanned by $\sigma_1, \dots, \sigma_m$. Let $P_{x,n}$ be the subset of the $(n-1)$ -sphere Σ described in Lemma 1. Thus for certain positive constants A and η , depending only on n and the parameter δ of the weak cone condition, $\lambda(P_{x,n}) = A$ for all $x \in \Omega$. Let ν denote $(m-1)$ -dimensional Lebesgue measure on the $(m-1)$ -sphere. The following lemma holds.

LEMMA 8. *There exist positive constants K_3, K_4 , and K_5 (depending on n, δ , and k), and for each $x \in \Omega \cap H$ there exists a subset A_x of $\mathcal{O}(n)$ such that*

- (i) $\mu(A_x) \geq K_3,$
- (ii) $|\det(\sigma_1, \sigma_2, \dots, \sigma_m, e_{m+1}, \dots, e_n)| \geq K_4 \quad \text{for all } \sigma \in A_x,$
- (iii) $\nu(P_{x,n} \cap E(\sigma, x)) \geq K_5 \quad \text{for } \sigma \in A_x.$

Assume, for the moment, that Lemma 8 is true. (We prove it shortly.) Let $x \in \Omega \cap H$, let $\sigma \in A_x$, and let $C_{x,n}$ be the generalized cone of Lemma 1. Condition (iii) guarantees that $C_{x,n} \cap E(\sigma, x)$ is itself a generalized cone (m -dimensional) with the same value of η and with m -volume bounded away from zero by a constant depending only on n, δ, k . By (2) (in Lemma 2) applied in the m -plane $E(\sigma, x)$, we have for all $x \in \Omega$, and all $\sigma \in A_x$

$$|u(x)| \leq K_6(n, \delta, k) \|u\|_{[m,1,E(\sigma,x)]}.$$

Integrating σ over A_x , and denoting by dS_σ the Lebesgue volume element in $E(\sigma, x)$, we obtain

$$\begin{aligned} |u(x)| &\leq \frac{K_6}{K_3} \int_{A_x} d\mu(\sigma) \int_{E(\sigma, x)} \sum_{|\alpha| \leq m} |D^\alpha u| dS_\sigma \\ &\leq K_7 \int_{T'} d\mu(\sigma) \int_{E(\sigma, x)} \sum_{|\alpha| \leq m} |D^\alpha u| dS_\sigma, \end{aligned}$$

where $K_7 = K_6/K_3$, and $T' = \{\sigma \in \mathcal{O}(n): \text{inequality (ii) holds}\}$. Hence

$$\begin{aligned} \int_{\Omega \cap H} |u(x)| dx'' &\leq K_7 \int_{T'} d\mu(\sigma) \int_H dx'' \int_{E(\sigma, x)} \sum_{|\alpha| \leq m} |D^\alpha u| dS_\sigma \\ &\leq K_7 \int_{T'} \frac{d\mu(\sigma)}{\det(\sigma_1, \dots, \sigma_m, e_{m+1}, \dots, e_n)} \int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u(x)| dx \\ &\leq (K_7/K_4) \|u\|_{m,1,\Omega} \end{aligned}$$

which is the desired imbedding inequality. ■

Proof of Lemma 8. Let Σ_m denote the $(m-1)$ -sphere in \mathbb{R}^m . For $x \in \Omega \cap H$ we have

$$\begin{aligned} \int_{\mathcal{O}(n)} \nu(P_{x,\eta} \cap E(\sigma, x)) d\mu(\sigma) &= \int_{\mathcal{O}(n)} d\mu(\sigma) \int_{\Sigma_m} 1_{P_{x,\eta}} \left(\sum_{j=1}^n s_j \sigma_j \right) ds \\ &= \int_{\Sigma_m} \mu \left(\left\{ \sigma \in \mathcal{O}(n) : \sum_{j=1}^m s_j \sigma_j \in P_{x,\eta} \right\} \right) ds \\ &= (\nu(\Sigma_m)/\lambda(\Sigma)) \lambda(P_{x,\eta}) = 4K_3\nu(\Sigma_m) \quad (\text{say}). \end{aligned}$$

(Note that the integrand in the last integral above is actually independent of s .) Let $B_x = \{\sigma \in \mathcal{O}(n): \nu(P_{x,\eta} \cap E(\sigma, x)) \geq K_5\}$ where $K_5 = 2K_3\nu(\Sigma_m)$. Then

$$\begin{aligned} 4K_3\nu(\Sigma_m) &= \left(\int_{B_x} + \int_{\mathcal{O}(n) \sim B_x} \right) \nu(P_{x,\eta} \cap E(\sigma, x)) d\mu(\sigma) \\ &\leq \nu(\Sigma_m) \mu(B_x) + 2K_3\nu(\Sigma_m) \mu(\mathcal{O}(n)) \end{aligned}$$

so that $\mu(B_x) \geq 2K_3$.

Since $\mu(\{\sigma \in \mathcal{O}(n): \det(\sigma_1, \dots, \sigma_m, e_{m+1}, \dots, e_n) = 0\}) = 0$ there exists a constant $K_4 = K_4(n, k)$ such that

$$\mu(\{\sigma \in \mathcal{O}(n): |\det(\sigma_1, \dots, \sigma_m, e_{m+1}, \dots, e_n)| < K_4\}) \leq K_3.$$

Hence we may set $A_x = \{\sigma \in B_x: |\det(\sigma_1, \dots, \sigma_m, e_{m+1}, \dots, e_n)| \geq K_4\}$, and obtain the three properties asserted in the lemma. ■

6. PROOFS OF THE INTERPOLATION THEOREMS

Proof of Theorem 2. We wish to establish the inequality

$$\|u\|_{j,p} \leq K\{\epsilon \|u\|_{m,p} + \epsilon^{-j/(m-j)} \|u\|_p\} \quad (14)$$

for every ϵ in the interval $(0, \epsilon_0]$ and all $j \leq m-1$. Without loss of generality we may assume $\epsilon_0 = 1$ (otherwise replace ϵ by ϵ/ϵ_0 and suitably adjust K). We may also restrict our attention to the special case $j = 1$, $m = 2$, that is

$$\|u\|_{1,p} \leq K\{\epsilon \|u\|_{2,p} + \epsilon^{-1} \|u\|_p\} \quad (15)$$

for if (15) holds for all ϵ , $0 < \epsilon \leq 1$, and all $u \in W^{2,p}(\Omega)$ then (14) follows for all such ϵ , and all $u \in W^{m,p}(\Omega)$ (with a new constant K) by a double induction argument on j and m . (First perform an induction on m with $j = m-1$ and then a downward induction on j for fixed m . The details may be found in Adams [1, p. 73].)

We begin by obtaining a one-dimensional inequality. Let $f \in C^2([0, 1])$, let $0 < x < \frac{1}{3}$, and let $\frac{2}{3} < y < 1$; then there exists $z \in (x, y)$ such that

$$|f'(z)| = \left| \frac{f(y) - f(x)}{y - x} \right| \leq 3 |f(x)| + 3 |f(y)|.$$

Thus,

$$\begin{aligned} |f'(0)| &= \left| f'(z) - \int_0^z f''(t) dt \right| \\ &\leq 3 |f(x)| + 3 |f(y)| + \int_0^1 |f''(t)| dt. \end{aligned}$$

We integrate x over $(0, \frac{1}{3})$ and y over $(\frac{2}{3}, 1)$ to obtain

$$\frac{1}{9} |f'(0)| \leq \int_0^{1/3} |f(x)| dx + \int_{2/3}^1 |f(y)| dy + \frac{1}{9} \int_0^1 |f''(t)| dt,$$

and so, by Hölder's inequality,

$$|f'(0)|^p \leq K_p \left\{ \int_0^1 |f''(t)|^p dt + \int_0^1 |f(t)|^p dt \right\}$$

where $K_p = 2^{p-1} \cdot 9^p$. By a change of variable, we have, for $g \in C^2([0, \xi])$, that

$$|g'(0)|^p \leq \frac{K_p}{\xi} \left\{ \xi^p \int_0^\xi |g''(t)|^p dt + \xi^{-p} \int_0^\xi |g(t)|^p dt \right\}. \quad (16)$$

Now suppose $u \in C^\infty(\Omega)$, and for each $x \in \Omega$ let $P_{x,\eta}$ be the subset of Σ described in Lemma 1, so that $\lambda(P_{x,\eta}) = A(n, \delta) > 0$. If $0 < \xi < \eta$ and $\sigma \in P_{x,\eta}$, we have, by (16), that

$$|\sigma \cdot \text{grad } u(x)|^p \leq (K_p/\xi) I(\xi, p, u, x, \sigma),$$

where

$$I(\xi, p, u, x, \sigma) = \xi^p \int_0^\xi \left| \frac{d^2}{dt^2} u(x + t\sigma) \right|^p dt + \xi^{-p} \int_0^\xi |u(x + t\sigma)|^p dt.$$

Clearly, there exists a positive constant $K = K(n, p, \delta)$ such that, for each $x \in \Omega$,

$$\int_{P_{x, \eta}} |\sigma \cdot \text{grad } u(x)|^p d\sigma \geq K |\text{grad } u(x)|^p.$$

Thus

$$|\text{grad } u(x)|^p \leq \frac{K_p}{K_\xi} \int_\Sigma I(\xi, p, u, x, \sigma) d\sigma$$

and

$$\int_\Omega |\text{grad } u(x)|^p dx \leq \frac{K_p}{K_\xi} \int_\Sigma d\sigma \int_\Omega I(\xi, p, u, x, \sigma) dx.$$

In order to estimate the inner integral on the right, we again regard u and its derivatives as extended to all of \mathbb{R}^n so as to vanish identically outside Ω . For simplicity, we suppose $\sigma = e_n$ and so, setting $x' = (x_1, \dots, x_{n-1})$, obtain

$$\begin{aligned} & \int_\Omega I(\xi, p, u, x, e_n) dx \\ &= \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^\infty dx_n \int_0^\xi \{ \xi^p |D_n^2 u(x', x_n + t)|^p + \xi^{-p} |u(x', x_n + t)|^p \} dt \\ &= \int_{\mathbb{R}^{n-1}} dx' \int_0^\xi dt \int_{-\infty}^\infty \{ \xi^p |D_n^2 u(x)|^p + \xi^{-p} |u(x)|^p \} dx_n \\ &= \xi \int_\Omega \{ \xi^p |D_n^2 u(x)|^p + \xi^{-p} |u(x)|^p \} dx. \end{aligned}$$

In general, for $\sigma \in \Sigma$, we have

$$\int_\Omega I(\xi, p, u, x, \sigma) dx \leq \xi \{ \xi^p \|u\|_{2,p}^p + \xi^{-p} \|u\|_p^p \},$$

and so

$$\begin{aligned} \|u\|_{1,p}^p &\leq (nK_p/K) \int_\Sigma d\sigma \{ \xi^p \|u\|_{2,p}^p + \xi^{-p} \|u\|_p^p \} \\ &\leq (nK_p/K) \lambda(\Sigma) \{ \xi^p \|u\|_{2,p}^p + \xi^{-p} \|u\|_p^p \}. \end{aligned}$$

Inequality (15) now follows if we take p th roots and set $\xi = \eta\epsilon$. Since $C^\infty(\Omega) \cap W^{2,p}(\Omega)$ is dense in $W^{2,p}(\Omega)$, the proof is complete. ■

Proof of Theorem 3. We deal separately with the two cases $mp < n$ and $mp \geq n$. If $mp < n$, then $W^{m,p}(\Omega) \rightarrow L^{p^*}(\Omega)$, where $p^* = np/(n - mp)$. If $p \leq q \leq p^*$, we have, by Hölder's inequality and this imbedding,

$$\|u\|_q \leq \|u\|_{p^*}^\theta \|u\|_p^{1-\theta} \leq K \|u\|_{m,p}^\theta \|u\|_p^{1-\theta}, \quad (17)$$

where $\theta = (n/mp) - (n/mq)$.

If either $mp = n$ and $p \leq q < \infty$, or $mp > n$ and $p \leq q \leq \infty$, we proceed as follows to obtain (17). Let $u \in C^\infty(\Omega)$, and let η be as in Lemma 1. For $x \in \Omega$ and $0 < \xi \leq \eta$, we obtain, by Lemma 2, (regarding, as in Section 4, u and its derivatives as extended to vanish identically outside Ω , and setting $\omega_m(x) = |x|^{m-n}$),

$$|u(x)| \leq K_1 \left\{ \sum_{|\alpha| \leq m-1} \xi^{|\alpha|-n} 1_{B_\xi} * |D^\alpha u|(x) + \sum_{|\alpha|=m} 1_{B_\xi} \omega_m * |D^\alpha u|(x) \right\}. \quad (18)$$

We estimate the L^q -norm of each term on the right side of (18) by Young's inequality for convolution; if $1/s = 1 - (1/p) + (1/q)$, then

$$\|1_{B_\xi} * |D^\alpha u|\|_q \leq \|1_{B_\xi}\|_s \|D^\alpha u\|_p = K_2 \xi^{n-(n/p)+(n/q)} \|D^\alpha u\|_p$$

and

$$\|1_{B_\xi} \omega_m * |D^\alpha u|\|_q \leq \|1_{B_\xi} \omega_m\|_s \|D^\alpha u\|_p = K_3 \xi^{m-(n/p)+(n/q)} \|D^\alpha u\|_p.$$

(Note that K_3 is finite by virtue of the restrictions placed on q .) Hence we obtain from (18)

$$\|u\|_q \leq K_4 \left\{ \sum_{j=0}^{m-1} \xi^{j-(n/p)+(n/q)} \|u\|_{j,p} + \xi^{m-(n/p)+(n/q)} \|u\|_{m,p} \right\}.$$

But, by Theorem 2,

$$\|u\|_{j,p} \leq K_5 \{ \xi^{m-j} \|u\|_{m,p} + \xi^{-j} \|u\|_p \},$$

and we have

$$\|u\|_q \leq K_6 \{ \xi^{-(n/p)+(n/q)} \|u\|_p + \xi^{m-(n/p)+(n/q)} \|u\|_{m,p} \}. \quad (19)$$

Inequality (19) may be asserted to hold for all $\xi \leq 1$ provided K_6 is suitably modified. Choosing ξ so that the two terms on the right side of (19) are equal we obtain (17). By a density argument, (17) holds for all $u \in W^{m,p}(\Omega)$. ■

Proof of Theorem 4. Suppose that $(m-1)p < n < mp$, and $q \geq 1$. It is sufficient to show that the inequality

$$|u(x)| \leq K \|u\|_{m,p}^\theta \|u\|_q^{1-\theta}, \quad \theta = np/(np + (mp - n)q) \quad (20)$$

holds for all $x \in \Omega$ and all $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$.

First let us note that (20) is a straightforward consequence of Theorem 3 and Hölder's inequality, if $1 \leq q \leq p$. Indeed, we do not need the assumption $(m-1)p < n$ in this case, for, if $n < mp$, we have by Theorem 3, that

$$\|u\|_p \leq K \|u\|_{m,p}^{n/mp} \|u\|_p^{1-(n/mp)},$$

and by Hölder's inequality

$$\|u\|_p \leq \|u\|_q^{q/p} \|u\|_\infty^{1-(q/p)},$$

whence (20) follows by substitution and cancellation.

Now suppose that $q \geq p$, and, for the moment, that $m = 1$ and $p > n$. We have, from (18), that

$$|u(x)| \leq K_1 \left\{ \xi^{-n} 1_{B_\xi} * |u|(x) + \sum_{|\alpha|=1} 1_{B_\xi} \omega_1 * |D^\alpha u|(x) \right\},$$

for $0 < \xi \leq \eta$. By Hölder's inequality,

$$1_{B_\xi} * |u|(x) \leq K_2 \xi^{n-(n/q)} \|u\|_q,$$

and, for $|\alpha| = 1$,

$$1_{B_\xi} \omega_1 * |D^\alpha u|(x) \leq K_3 \xi^{1-(n/p)} \|D^\alpha u\|_p.$$

Hence

$$|u(x)| \leq K_4 \{ \xi^{-n/q} \|u\|_q + \xi^{1-(n/p)} \|u\|_{1,p} \}. \quad (21)$$

Since $\|u\|_q \leq K_5 \|u\|_{1,p}$ holds for some constant K_5 , and since (21) may be asserted to hold for all ξ with $0 < \xi^{1-(n/p)+(n/q)} \leq K_5$, provided K_4 is suitably modified, we may choose ξ to make the two terms on the right side of (21) equal. This choice yields (20) with $m = 1$.

For general m , we have $W^{m,p}(\Omega) \rightarrow W^{1,r}(\Omega)$, where $r = np/(n - mp + p)$ satisfies $n < r < \infty$, since $(m-1)p < n < mp$. Hence, if $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$, we have

$$|u(x)| \leq K_6 \|u\|_{1,r}^\theta \|u\|_q^{1-\theta} \leq K_7 \|u\|_{m,p}^\theta \|u\|_q^{1-\theta},$$

where $\theta = nr/[nr + (r-n)q] = np/[np + (mp-n)q]$. ■

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